Math 3235 Probability Theory

09/15/22

Expectation of a r.v.

X $\qquad p_X(x)$ p.m.f.

$$\mathbb{E}(X) = \sum_x x\, p_X(x)$$

Function of a r.v.

X is a r.v.

$f : \mathbb{R} \to \mathbb{R}$

Then

$Y = f(X)$ is a r.v. itself.

Since X is discrete, we need no conditions on f.

$$p_Y(y) = \sum_{x \mid f(x)=y} p_X(x) = \sum_{x \in f^{-1}(y)} p_X(x)$$

$$x \mid f(x) = y \iff x \in f^{-1}(y)$$

Th: if $Y = f(X)$ where x is a discrete r.v. Then

$$\mathbb{E}(f(X)) = \sum_x f(x)\, p_X(x)$$

Proof

$$\mathbb{E}(Y) = \sum_y y\, p_Y(y) =$$

$$= \sum_y y \sum_{x \in f^{-1}(y)} p_X(x) =$$

$$= \sum_y \sum_{x \in f^{-1}(y)} f(x)\, p_X(x) =$$

$$= \sum_x f(x)\, p_X(x).$$

Q.E.D.

Poisson r.v. X par. λ

$Y = e^X$

$$\mathbb{E}(Y) = \sum_x e^x \frac{\lambda^x}{x!} e^{-\lambda} =$$

$$= \sum_x \frac{(e\lambda)^x}{x!} e^{-\lambda} = e^{\lambda(e-1)}$$

Th: X is a r.v.

a) if $\mathbb{P}(X \geq 0) = 1$
$\mathbb{E}(X) = 0$ $\implies$ $\mathbb{P}(X = 0) = 1$

b) $\mathbb{E}(aX + b) = a\,\mathbb{E}(X) + b$

Proof

b) $\mathbb{E}(aX + b) = \sum_x (ax + b)\, p(x) =$

$= a \sum_x x\, p(x) + b \sum_x p(x) =$

$= a\,\mathbb{E}(X) + b$

a) if $\mathbb{P}(X \geq 0) = 1 \implies$

$x \in \operatorname{Im}(X) \implies x \geq 0$

$\mathbb{E}(X) = 0 \implies x\, \mathbb{P}(X = x) = 0$
$\forall x > 0$

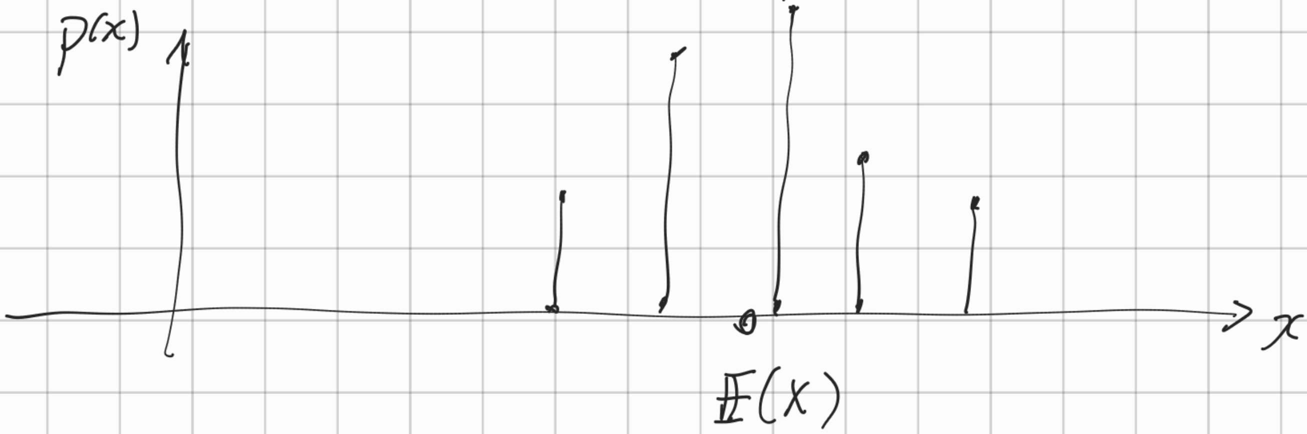
$\mathbb{P}(X = 0) = 1$

Variance.

$$V(X) = var(X) =$$

$$\mathbb{E}\left(\left(X - \mathbb{E}(X)\right)^2\right)$$

Expected value: measure of position.



Measure of variability.

How distant from the Exp. val.

is a "normal" value of the r.v.,

in average?

$$\mathbb{E}\left(X - \mathbb{E}(X)\right) =$$

$$\mathbb{E}(X) - \mathbb{E}(X) = 0$$

Not very useful!!

$$\boxed{\mathbb{E}\left(\left(X - \mathbb{E}(X)\right)^2\right) = var(X)}$$

$\sigma_X = \sqrt{var(X)}$ standard deviation.

$var(X) \geq 0$

$var(X) = 0 \Rightarrow \mathbb{P}(X - \mathbb{E}(X) = 0) = 1$

$X = \mathbb{E}(X)$ with prob. 1.

This means that X Takes a unique value with prob. 1.

How To compute variances?

$$var(X) = \sum_x (x - \mathbb{E}(X))^2 p(x) =$$

$$\sum_x (x^2 - 2\mu x + \mu^2) p(x) =$$

where we set $\mathbb{E}(X) = \mu$.

$$\sum_x x^2 p(x) - 2\mu \sum_x x p(x) + \mu^2 \sum_x p(x) =$$

$$\mathbb{E}(X^2) - 2\mu \mathbb{E}(X) + \mu^2$$

$$\boxed{\mathrm{var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2}$$

since $\mathrm{var}(X) \geq 0$

$$\boxed{\mathbb{E}(X^2) \geq \mathbb{E}(X)^2}$$

$$\mathrm{var}(aX + b) = ?$$

$$\mathrm{var}(aX) = \mathbb{E}(a^2 X^2) - \mathbb{E}(aX)^2 =$$

$$= a^2 \, \mathrm{var}(X)$$

$$\mathrm{var}(X + b) = \mathbb{E}\left(\left(X + b - \mathbb{E}(X+b)\right)^2\right) =$$

but $\mathbb{E}(X+b) = \mathbb{E}(X) + b$

$$\mathrm{var}(X + b) = \mathrm{var}(X)$$

$$\boxed{\mathrm{var}(aX + b) = a^2 \, \mathrm{var}(X)}$$

$$Y = aX + b$$

$$\boxed{\sigma_Y = |a| \, \sigma_X}$$

Example

$$\sigma_{-X} = \sigma_X$$

Examples.

Bernoulli p

$$\mathbb{E}(X) = p$$

$$\mathrm{var}(X) = p - p^2 = pq$$

$\mathrm{var}(X)$ is minimal for $p = 0$ or $p = 1$

$\mathrm{var}(X)$ is maximal for $p = 0.5$

Binomial r.v. with par N, p

$$\mathbb{E}(X^2) = \mathbb{E}(X(X-1)) + \mathbb{E}(X)$$

$$\mathrm{var}(X) = \mathbb{E}(X(X-1)) + \mathbb{E}(X) - \mathbb{E}(X)^2$$

$$= \mathbb{E}(X(X-1)) - \mathbb{E}(X)(\mathbb{E}(X)-1)$$

$$\mathbb{E}(X(X-1)) = \sum_x x(x-1)\binom{N}{x} p^x q^{N-x}$$

$$x(x-1)\binom{N}{x} = \frac{N!}{x!(N-x)!}x(x-1) =$$

$$= \frac{N!}{(x-2)!(N-x)!} =$$

$$= N(N-1)\frac{(N-2)!}{(x-2)!(N-x)} =$$

$$= N(N-1)\binom{N-2}{x-2}$$

Thus

$$\sum_{x=0}^{N} x(x-1)\binom{N}{x} p^x q^{N-x} =$$

$$N(N-1)\sum_{x=2}^{N}\binom{N-2}{x-2} p^x q^{N-x} =$$

$z = x-2$

$$N(N-1)p^2\sum_{z=0}^{N-2}\binom{N-2}{z} p^z q^{N-2-z} = N(N-1)p^2$$

$$\text{var}(X) = N(N-1)p^2 - Np(Np-1) =$$

$$= Np(1-p)$$

X is Poisson par λ

$$\mathbb{E}(X(X-1)) = \sum_{x \geq 0} x(x-1) \frac{\lambda^x}{x!} e^{-\lambda} =$$

$$= \lambda^2 \sum_{x \geq 2} \frac{\lambda^{x-2}}{(x-2)!} e^{-\lambda} =$$

$$= \lambda^2$$

$$\operatorname{var}(X) = \mathbb{E}(X(X-1)) + \mathbb{E}(X) - \mathbb{E}(X)^2 =$$

$$= \lambda$$

OTher way

$$N p (1-p) \quad \text{with} \quad p = \frac{\lambda}{N}$$

$$\downarrow$$

$$\lambda \left(1 - \frac{\lambda}{N}\right) \xrightarrow[N \to \infty]{} \lambda$$

X geometric p

$$\mathbb{E}(X) = \frac{1}{p}$$

$$\mathbb{E}(X(X-1)) = pq \sum_{x \geq 0} x(x-1)\, q^{x-2} =$$

$$= pq \frac{d^2}{dq^2} \left(\sum_{x \geq 2} q^x \right) =$$

$$= \frac{2q}{(1-q)^3} = \frac{2q}{p^3}$$

$$\mathrm{var}(X) = q\, p^{-2}$$